JOURNAL OF ECONOMIC THEORY 18, 216–223 (1978)

Notes, Comments, and Letters to the Editor

A Note on Efficient Growth with Irreversible Investment and the Phelps–Koopmans Theorem*

1. Introduction

An interesting problem in the theory of efficient allocation of resources over time is to formulate suitable criteria that can completely characterize the set of efficient programs.

Restricting our attention to the standard aggregative model of economic growth, we find that, beginning with the partial characterization result of Phelps [12], this problem has inspired detailed investigation in several papers by McFadden [8], Cass [7], Benveniste and Gale [5], Benveniste [4], and Mitra [9]. An important assumption in all these papers is that, in any time period, the existing amount of the single (capital cum consumption) good, whether in the form of current output or depreciated capital stock, can be totally consumed. In other words, it is feasible to run down the capital stock at any rate we wish and to enjoy a corresponding increase in consumption.

It is, however, clearly sensible to argue that investments, once made in physical form, *cannot* be converted readily into consumption. Hence, investment should be "irreversible." Of course, we should allow for the possibility that capital can, *within limits*, be run down to permit more consumption; namely, capital depreciates and failure to replace it constitutes a method of increasing consumption at the expense of capital. A reasonable assumption, then, taking account of the above arguments, is that *gross investment* (i.e., the net increase in capital plus the amount of depreciation) be nonnegative at each point of time. (*Net investment* can be negative to the extent of depreciation, but no more.)

Such a restriction on investment has, of course, received some attention in the literature on normative growth theory. The implication of this restriction for programs, optimal with regard to final stocks in a finite-horizon planning framework, has been studied by Solow [13], and programs, optimal

* This research was partially supported by National Science Foundation Grant SOC 76-14342 to Cornell University.

0022-0531/78/0181–0216$02.00/0

with regard to a Ramsey-type utility functional in an infinite-horizon framework by Cass [6], Arrow [1], and Arrow and Kurz [2, 3]. However, in the literature on efficient growth, this restriction on investment has been virtually ignored. The present study incorporates the assumption of irreversibility of investment in the standard aggregative model and investigates its implications on the nature of efficient programs.

First, an example is constructed to show that even if a program always has a capital stock above, and bounded away from, the golden-rule stock, it can still be efficient (see Sect. 3). This means that the well-known Phelps–Koopmans result[1] no longer holds.

Second, a set of necessary and sufficient conditions for the inefficiency of a feasible program is presented in Section 4. It is shown that a feasible program, satisfying a "smoothness condition," is *inefficient* if and only if (a) the value of investment is bounded away from zero, from a certain time onward, *and* (b) the ratio of the share of primary factor to the value of capital deteriorates too fast. For a precise statement of the result, see Theorem 4.1; for precise definitions of the terms used, see Section 2.

2. The Model

Consider a one-good economy, with a technology given by a function g, from R^+ to itself. The production possibilities consist of capital input x, and current output $y = g(x)$, for $x \geqslant 0$.

The following assumptions on g are used:

(A.1) $g(0) = 0$.

(A.2) g is strictly increasing for $x \geqslant 0$.

(A.3) g is concave for $x \geqslant 0$.

(A.4) g is differentiable for $x \geqslant 0$.

The initial capital input, $\mathbf{x}$, is considered to be historically given and positive. Capital stock is considered to depreciate at a constant rate, d, where $0 < d \leqslant 1$, whether it is used for production or not.

[1] This result was conjectured by Phelps [11], and its proof, based on a proof provided by Koopmans, appeared in Phelps [12]. It is, therefore, known in the literature as the Phelps–Koopmans theorem. It states that if the capital stock of a program is above, and bounded away from, the golden-rule stock, from a certain time onward, then the program is inefficient. No nonnegativity restriction on investment is imposed in the model in which this result is obtained. For a precise statement, see either Phelps [12] or Cass [7].

A *feasible program* is a sequence $(x, y, z, c) = (x_t, y_{t+1}, z_{t+1}, c_{t+1})$ satisfying

$$x_0 = \mathbf{x}, \qquad y_{t+1} = g(x_t) \qquad \text{for} \quad t \geqslant 0,$$
$$z_t = x_t - (1-d)\, x_{t-1}, \qquad c_t = y_t - z_t \qquad \text{for} \quad t \geqslant 1, \tag{2.1}$$
$$x_t, y_t, z_t, c_t \geqslant 0 \qquad \text{for} \quad t \geqslant 1.$$

(z is to be interpreted as gross investment; c as consumption.[2]) A feasible program (x', y', z', c') *dominates* a feasible program (x, y, z, c) if $c_t' \geqslant c_t$ for all $t \geqslant 1$, and $c_t' > c_t$ for some t. A feasible program (x, y, z, c) is *inefficient* if there is a feasible program (x', y', z', c') which dominates it. A feasible program is *efficient* if it is not inefficient.

Define a *total output function* f by

$$f(x) = g(x) + (1-d)\, x \qquad \text{for} \quad x \geqslant 0 \tag{2.2}$$

The *share of primary factor in total output* is defined by

$$W(x) = 1 - \{f'(x)\, x/f(x)\} \qquad \text{for} \quad x > 0; \qquad W(x) = 0 \qquad \text{for} \quad x = 0. \tag{2.3}$$

The *competitive price sequence* $p = (p_t)$ associated with a feasible program (x, y, z, c) is given by

$$p_0 = 1, \qquad p_{t+1} = p_t/f'(x_t) \qquad \text{for} \quad t \geqslant 0 \tag{2.4}$$

These are precisely the prices which yield maximum *intertemporal profits* for each t; that is,

$$p_{t+1} f(x_t) - p_t x_t \geqslant p_{t+1} f(x) - p_t x, \qquad x \geqslant 0, \quad t \geqslant 0. \tag{2.5}$$

The *value of capital* sequence $v = (v_t)$ associated with a feasible program (x, y, z, c) is given by

$$v_t = p_t x_t \qquad \text{for} \quad t \geqslant 0. \tag{2.6}$$

[2] It should be noted that the present framework includes the neoclassical growth model as a special case, by a suitable interpretation of variables. Let $G(X, L)$ be a constant returns to scale production function defined on Capital (X) and Labor (L). Labor is assumed to grow exogenously at a rate $n \geqslant 0$, i.e., $L_t = L_0(1 + n)^t$ for $t \geqslant 0$; $L_0 > 0$. Capital depreciates at a constant rate $\hat{d}$, where $0 < \hat{d} \leqslant 1$. The basic neoclassical growth equation is $Y_{t+1} = G(X_t, L_t) = C_{t+1} + X_{t+1} - (1 - \hat{d})X_t$. Dividing through by L_{t+1}, and denoting $[Y_{t+1}/L_{t+1}]$ by y_{t+1}, $[X_t/L_t]$ by x_t, $[C_{t+1}/L_{t+1}]$ by c_{t+1}, and $[Z_{t+1}/L_{t+1}]$ by z_{t+1} for $t \geqslant 0$, we have $y_{t+1} = [G(x_t, 1)/(1 + n)] = c_{t+1} + z_{t+1} = c_{t+1} + (x_{t+1} - [(1 - \hat{d})/(1 + n)]x_t)$. Then letting $g(x) = G(x, 1)/(1 + n)$, and $d = (n + \hat{d})/(1 + n)$, we obtain the framework described in Section 2.

The corresponding *value of investment* sequence $u = (u_t)$ is given by

$$u_t = p_t z_t \qquad \text{for} \quad t \geqslant 1 \tag{2.7}$$

A nonnegative sequence (k_t) is said to be *bounded away from zero* if $\inf_{t\geqslant 0} k_t > 0$; it is said to be *bounded away from zero from a certain time onward* if $\liminf_{t\to\infty} k_t > 0$; it is said to *deteriorate too fast* if $\sum_{t=0}^{\infty} k_t < \infty$.

3. Discussion of Inefficiency and the Phelps–Koopmans Result

A convenient starting point of a discussion of inefficiency in this framework, is the following result.

Proposition 3.1. *Under* (A.1)–(A.4), *if a feasible program* (x, y, z, c) *is inefficient,*

$$\liminf_{t\to\infty} p_t z_t > 0. \tag{3.1}$$

Proof. If (x, y, z, c) is inefficient, there is a feasible program (x', y', z', c') which dominates it. Hence, we have

$$0 \leqslant c'_{t+1} - c_{t+1} = (y'_{t+1} - y_{t+1}) - (z'_{t+1} - z_{t+1}) \qquad \text{for} \quad t \geqslant 0. \tag{3.2}$$

By (2.1), $c_{t+1} = y_{t+1} - z_{t+1} = g(x_t) - \{x_{t+1} - (1-d)\, x_t\} = f(x_t) - x_{t+1}$, and $c'_{t+1} = f(x'_t) - x'_{t+1}$, for $t \geqslant 0$. Hence, we obtain

$$c'_{t+1} - c_{t+1} = [f(x'_t) - f(x_t)] - (x'_{t+1} - x_{t+1}) \qquad \text{for} \quad t \geqslant 0. \tag{3.3}$$

From (3.2) and (3.3), we have

$$(x_{t+1} - x'_{t+1}) \geqslant f(x_t) - f(x'_t) \qquad \text{for} \quad t \geqslant 0 \tag{3.4}$$

Let t_t be the first period when $c'_{t_1} > c_{t_1}$. Then, by (3.3), $x_t = x'_t$ for $t \leqslant t_1 - 1$, and $x'_{t_1} > x_{t_1}$, and by (3.4), $x_t > x'_t$ for $t \geqslant t_1$. Define $e_t = (x_t - x'_t)$ for $t \geqslant 0$. Then, $e_t = 0$ for $t \leqslant t_1 - 1$, and $e_t > 0$ for $t \geqslant t_1$. Also, by (A.3), (A.4), and (3.4), $(x_{t+1} - x'_{t+1}) \geqslant f'(x_t)(x_t - x'_t)$ for $t \geqslant 0$, so that by (2.4), $p_{t+1}e_{t+1} \geqslant p_t e_t$ for $t \geqslant 0$; that is, $p_t e_t \geqslant p_{t_1} e_{t_1} > 0$, for $t \geqslant t_1$. By (3.2), we have $(z_{t+1} - z'_{t+1}) \geqslant (y_{t+1} - y'_{t+1}) = g(x_t) - g(x'_t)$, for $t \geqslant 0$. By (A.3), (A.4), $(z_{t+1} - z'_{t+1}) \geqslant g'(x_t)(x_t - x'_t)$ for $t \geqslant 0$. By (2.4), $p_{t+1}(z_{t+1} - z'_{t+1}) \geqslant [g'(x_t)/f'(x_t)]\, p_t(x_t - x'_t)$ for $t \geqslant 0$. Hence, for $t \geqslant t_1$,

$$p_{t+1}(z_{t+1} - z'_{t+1}) \geqslant [g'(x_t)/f'(x_t)]\, p_t e_t \geqslant [g'(x_t)/f'(x_t)]\, p_{t_1} e_{t_1} \tag{3.5}$$

Now, we claim that $[g'(x_t)/f'(x_t)]$ is bounded away from zero. Suppose, on the contrary, that $[g'(x_t)/f'(x_t)] \to 0$, along a subsequence of t. Then, since $[g'(x_t)/f'(x_t)] = [1/\{1 + [(1-d)/g'(x_t)]\}]$, so $g'(x_t)$ must tend to zero for this subsequence. This means that $x_t \to \infty$ for this subsequence of t. However, for this subsequence, $f'(x_t) = g'(x_t) + (1-d) \to (1-d) < 1$, so there exists $0 < \bar{x} < \infty$, such that $f(\bar{x}) = \bar{x}$; $x < f(x) < \bar{x}$ for $0 < x < \bar{x}$; $\bar{x} < f(x) < x$ for $\bar{x} < x$. Hence, x_t must remain bounded. This contradiction establishes the claim. Hence, by (3.5), there is $\hat{h} > 0$, such that $p_{t+1}(z_{t+1} - z'_{t+1}) \geqslant \hat{h}$ for $t \geqslant t_1$, which implies (3.1) immediately. ∎

In particular, Proposition 3.1 implies that if, along a feasible program, investment is zero for a subsequence of periods, then it must be efficient, *no matter how the capital stock behaves over time*. Keeping this fact in mind, it is simple to construct an example, which shows that even if the capital stock of a program, at each point of time, is above, and bounded away from, the golden-rule stock, the program can be efficient. Thus, the Phelps–Koopmans result does not hold in this model.

EXAMPLE 3.1. Let $g(x) = x^{1/4}$; $d = \frac{1}{2}$. Then, $f(x) = g(x) + (1-d)\,x = x^{1/4} + \frac{1}{2}x$. Then, there are uniquely determined numbers, x^* and $\hat{x}$, such that $0 < x^* < \hat{x} < \infty$, and $f'(x^*) = 1, f(\hat{x}) = \hat{x}$. The number x^* represents the golden-rule stock; $\hat{x}$ is the maximum sustainable stock. Clearly, $x^* = (\frac{1}{2})^{4/3}$ and $\hat{x} = (2)^{4/3}$.

Consider the sequence (x, y, z, c) given by: $x_0 = \mathbf{x} = 2(\frac{2}{3})^{4/3}$; $z_t = 0$, $x_t = (\frac{2}{3})^{4/3}$, $y_t = 2^{1/4}(\frac{2}{3})^{1/3}$, $c_t = 2^{1/4}(\frac{2}{3})^{1/3}$ for t odd; $z_t = (\frac{2}{3})^{1/3}$, $x_t = 2(\frac{2}{3})^{4/3}$, $y_t = (\frac{2}{3})^{1/3}$, $c_t = 0$ for t even. It is simple to check that (x, y, z, c) satisfies (2.1), and is, therefore, a feasible program. Since $z_t = 0$ for t odd, the program is efficient, by Proposition 3.1. Also, $x_t \geqslant (\frac{2}{3})^{4/3} > x^*$ for $t \geqslant 0$, so that the capital stock of the program is above, and bounded away from, the golden-rule stock, for all time.

4. CHARACTERISATION OF INEFFICIENCY

In view of Example 3.1, it should be clear that there is a significant difference in the nature of inefficiency in this model, compared to traditional ones, where no nonnegativity restriction on investment is assumed. Thus, we can expect that the necessary and sufficient conditions for inefficiency would be different from traditional ones too, in that they would involve a condition on the behavior of investment. This section is devoted to finding such a set of conditions.

To this end, we consider, following Mitra [9], the following "smoothness condition" on a feasible program (x, y, z, c):

Condition S. For some $0 < m \leqslant M < \infty$ and $0 < r \leqslant 1$,

$$m\theta W(x_t)/x_t \leqslant \{[f(x_t) - f(x_t - \theta)]/\theta f'(x_t)\} - 1 \leqslant M\theta W(x_t)/x_t\,,$$
$$\text{for} \quad 0 < \theta < rx_t\,, \quad t \geqslant 0.$$

Regarding the plausibility of Condition S, the following remarks are pertinent: (i) Condition S is satisfied by feasible programs, generated by the class of functions $g(x) = Ax^q + Bx$, where $A, B \geqslant 0$, and $0 < q < 1$. (ii) If f is twice continuously differentiable, strictly concave (with $f'' < 0$), and satisfies the end point conditions: $0 \leqslant f'(\infty) < 1 < f'(x) < \infty$, for some $x > 0$, then any feasible program whose capital input stocks are bounded away from zero, satisfies Condition S.[3] (iii) If f is twice differentiable, and there are positive numbers, N, N', Q, Q' such that $N \leqslant [f'(x)\,x/f(x)] \leqslant N'$, $Q \leqslant [-f''(x)\,x^2/f(x)] \leqslant Q'$ for $x \geqslant 0$, then any feasible program satisfies Condition S.[4]

THEOREM 4.1. *Under* (A.1)–(A.4), *a feasible program* (x, y, z, c) *satisfying Condition S, is inefficient if and only if*

$$\liminf_{t\to\infty} p_t z_t > 0 \tag{4.1}$$

and

$$\sum_{t=0}^{\infty} [W(x_t)/p_t x_t] < \infty. \tag{4.2}$$

Proof (*necessity*). If (x, y, z, c) is inefficient, then (4.1) is satisfied by Proposition 3.1. To establish (4.2), follow the method used to prove the necessity part of Theorem 1 in [9].[5]

(*Sufficiency*). Suppose (x, y, z, c) satisfies Condition S, and (4.1), (4.2). Then, there is $t_1 \geqslant 1$, and $h > 0$, such that $p_t z_t \geqslant h$ for $t \geqslant t_1$. Also, there is $D < \infty$, such that $\sum_{s=0}^{\infty} [W(x_s)/p_s x_s] = D$. Now, define a sequence (a_t) in the following way. Let $a_t = 0$ for $0 \leqslant t < t_1$; $a_{t_1} = \min(\frac{1}{2}/p_{t_1}DM, rh/4p_{t_1})$, and, for $t \geqslant t_1$,

$$(1/p_{t+1}a_{t+1}) = (1/p_{t_1}a_{t_1}) - M \sum_{s=t_1}^{t} [W(x_s)/p_s x_s]. \tag{4.3}$$

Then,

$$0 < a_t < rz_t \leqslant rx_t \qquad \text{for} \quad t \geqslant t_1 \tag{4.4}$$

[3] These are the assumptions used by Cass [7].

[4] These are the assumptions used by Benveniste and Gale [5]. Remarks (i), (ii), and (iii) are verified in detail in Mitra [9].

[5] The steps are spelt out, in detail, in Mitra [10, Theorem 4.1].

Now, following the method used to prove the sufficiency part of Theorem 1 in [9],[6] we get, for $t \geqslant 0$, $a_{t+1} \geqslant f(x_t) - f(x_t - a_t)$. Now, define a sequence (b_t) as follows: $b_t = 0$ for $0 \leqslant t < t_1$, $b_{t_1} = a_{t_1}$, and for $t \geqslant t_1$, $b_{t+1} = f(x_t) - f(x_t - b_t)$. Then, clearly, $0 < b_t \leqslant a_t < rz_t \leqslant rx_t \leqslant x_t$ for $t \geqslant t_1$. Now, we construct a sequence (x', y', z', c') as follows: $x_0' = \mathbf{x}$, $x_t' = x_t - b_t$, $z_t' = x_t' - (1-d)\, x_{t-1}'$ for $t \geqslant 1$; $y_t' = g(x_{t-1}')$, $c_t' = y_t' - z_t'$ for $t \geqslant 1$. To check feasibility, note that $x_t' \geqslant 0$, and so $y_t' \geqslant 0$ for $t \geqslant 1$. Also, $z_t' = x_t' - (1-d)\, x_{t-1}' = (x_t - b_t) - (1-d)(x_{t-1} - b_{t-1}) = z_t + (1-d)\, b_{t-1} - b_t \geqslant z_t - b_t \geqslant 0$, for $t \geqslant 1$. Finally, $c_t' = y_t' - z_t' = g(x_{t-1}') - [x_t' - (1-d)\, x_{t-1}'] = f(x_{t-1}') - x_t' = f(x_{t-1} - b_{t-1}) - (x_t - b_t) \geqslant f(x_{t-1}) - x_t = c_t$ for $t \geqslant 1$. Also, clearly, $c_{t_1}' > c_{t_1}$, so that (x', y', z', c') dominates (x, y, z, c), proving that (x, y, z, c) is inefficient. ∎

Remark. The condition (4.1) clearly implies that

$$\inf_{t \geqslant 0} p_t x_t > 0. \tag{4.5}$$

In Mitra [9], inefficiency in a model with no nonnegativity restriction on investment was characterized in terms of (4.5) and (4.2). It is clear, therefore, that the added restriction on investment in the present model is reflected in the stronger condition (4.1) which needs to be satisfied for a program to be inefficient.[7]

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[6] For details, the reader is referred to Mitra [10, Theorem 4.1].

[7] By putting $d = 1$, in the present framework, we obtain the mathematical form of the traditional model studied by Cass [7], Benveniste and Gale [5], and Mitra [9]. (This statement should not, of course, be interpreted to mean that in these models, capital is assumed to be completely nondurable). Hence, Theorem 4.1 is a generalization of the characterization theorems in the above papers, and those results can be obtained as corollaries of Theorem 4.1.

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Received: June 6, 1977

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